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On the Numerical Solutions for the Time-Fractional Telegraph Equation

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***Abstract* – Fractional differential equations have recently been applied in various area of engineering, science, finance, applied mathematics, bio-engineering and others.In this paper, an efficient numerical method for solving telegraph equation with fractional time derivative $\alpha (1<\alpha \le 2)$, is proposed. The fractional derivative is described in the Caputo sense. This technique is derived by expanding the required approximate solution as the elements of shifted Legendre polynomials. Using the operational matrix of the fractional derivative the problem can be reduced to a set of algebraic equations. From the computational point of view, the solution obtained by this method is in excellent agreement with those obtained by previous work in the literature and also it is efficient to use.**

***Keywords* – Time-Fractional Telegraph Equation, Legendre Polynomials, Caputo Derivative.**

I. INTRODUCTION

In recent years, there has been a great deal of interest in fractional differential equations. Historical summaries of the developments of fractional calculus can be found in Oldham and Spanier [1], Miller and Ross [2], Samko et al. [3] and Podlubny [4]. A number of numerical methods have been proposed for fractional differential equations [5, 6, 7, 8, 9, 10, 14]. Suspension flows are traditionally modelled by parabolic partial differential equations. Sometimes they can be better modelled by hyperbolic equations such as the telegraph equation, which have parabolic asymptotics. In particular the experimental data described in [11, 12] seem to be better modelled by the telegraph equation than by the heat equation. Some of the related mathematics was discussed in [12]. The time-fractional telegraph equations have recently been considered by many authors. Orsingher and Beghin [13] studied the fundamental solutions to time fractional telegraph equations of order 2α. They obtained the Fourier transforms of the solutions for any α and gave a representation of their inverses in terms of stable densities. The equation is used in modeling reaction–diffusion in various branches of engineering sciences and biological sciences by many researchers like Mohebbi and Dehaghan [15], El-Azab and El-Glamel [16], Gao and Chi [17], Recently, Das and Gupta [18] have solved the fractional hyperbolic PDE by using HAM. In this paper, the shifted Legendre spectral method is used to obtain the approximate solution of telegraph equation with the fractional time derivative and the results are presented graphically for different particular cases. We describe some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

Definition 1. Caputo’s definition of the fractional-order derivative is defined as

$$D^{\alpha} f(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt, \qquad n-1<\alpha\le n,\ n\in N,$$

where α is the order of the derivative and n is the smallest integer greater than α. For the Caputo’s derivative we have:

$$D^{\alpha} C = 0, \qquad C \text{ is a constant}, \tag{1}$$

$$D^{\alpha} x^{\beta} = \begin{cases} 0, & \text{for } \beta\in N_0 \text{ and } \beta<\lceil\alpha\rceil \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, & \text{for } \beta\in N_0 \text{ and } \beta\ge\lceil\alpha\rceil \end{cases} \tag{2}$$

We use the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α. Also $N=1,2,...$ and $N_0=0,1,2,...$. Recall that for $\alpha\in N$, the Caputo differential operator coincides with the usual differential operator of integer order. Our main aim is to generalize Legendre operational matrix to fractional calculus. The organization of this paper is as follows. In the next section we describe the basic formulation of shifted Legendre polynomials . Section 3 summarizes the application of shifted Legendre tau method to the solution of time-fractional telegraph equation. As a result,set of algebraic equations are formed and a solution of the considered problem is introduced. In Section 4, some comparisons and numerical results are given to clarify the method. Also a conclusion is given in Section 5.

II. SHIFTED LEGENDRE POLYNOMIALS

The well-known Legendre polynomials are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formulas:

$$p_0(z)=1, \qquad p_1(z)=z,$$

$$p_{i+1}(z)=\frac{2i+1}{i+1}zp_i(z)-\frac{i}{i+1}p_{i-1}(z), \quad i=1,2,....$$

In order to use these polynomials on the interval[0,1], we define the so called shifted Legendre polynomials by introducing the change of variable

$$z=2x-1, \qquad 0\le x\le 1.$$

The shifted Legendre polynomials in x are then obtained as follows:

$$p_0(x)=1, \qquad p_1(x)=2x-1,$$

$$p_{i+1}(x) = \frac{(2i+1)(2x-1)}{i+1} p_i(x) - \frac{i}{i+1} p_{i-1}(x), \quad i = 1,2,....$$

The analytic form of the shifted Legendre polynomial $p_i(x)$ of degree i given by

$$p_i(x) = \sum_{k=0}^{i} (-1)^{(i+k)} \frac{(i+k)!x^k}{(i-k)!k!^2}. \qquad (3)$$

Note that $p_i(0) = (-1)^i$ and $p_i(1) = 1$. The orthogonality condition is

$$\int_0^1 p_i(x) p_j(x) dx = \begin{cases} 0 & for\ i = j \\ \dfrac{1}{2i+1} & for\ \ i \neq j. \end{cases}$$

A function $y(x)$, square integrable in $[0,1]$, may be expressed in terms of the shifted Legendre polynomials as

$$y(x) = \sum_{j=0}^{\infty} c_j p_j(x),$$

where the coefficients c_j are given by

$$c_j = (2j+1)\int_0^1 y(x) p_j(x) dx, \quad j = 1,2,....$$

In practice, only the first (m + 1)-terms shifted Legendre polynomials are considered. Then we have

$$y_m(x) = \sum_{j=0}^{m} c_j p_j(x) = C^T \phi(x), \qquad (4)$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $\phi(x)$ are given by

$$C^T = [c_0,...,c_m], \quad \phi(x) = [p_0(x), p_1(x),..., p_m(x)]^T. \qquad (5)$$

The derivative of the vector $\phi(x)$ can be expressed by

$$\frac{d\phi}{dx} = D^{(1)} \phi(x),$$

where $D^{(1)}$ is the $(m+1)(m+1)$ operational matrix of derivative and for odd m given as

$$D = 2\begin{pmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & ... & 0 & 0 & 0 \\ . & . & . & . & ... & . & . & . \\ . & . & . & . & ... & . & . & . \\ . & . & . & . & ... & . & . & . \\ 0 & 3 & 0 & 7 & ... & 2m-3 & 0 & 0 \\ 1 & 0 & 5 & 0 & ... & 0 & 2m-1 & 0 \end{pmatrix}$$

and for even m given as

$$D = 2\begin{pmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & ... & 0 & 0 & 0 \\ . & . & . & . & ... & . & . & . \\ . & . & . & . & ... & . & . & . \\ . & . & . & . & ... & . & . & . \\ 1 & 0 & 5 & 0 & ... & 2m-3 & 0 & 0 \\ 0 & 3 & 0 & 7 & ... & 0 & 2m-1 & 0 \end{pmatrix}$$

It is clear that

$$\frac{d^n\phi}{dx^n} = (D^{(1)})^n \phi(x),$$

where $n \in N$ and the superscript, in D^1 denotes matrix powers.Then

$$D^n = (D^{(1)})^n \quad n = 1,2,.... \qquad (6)$$

Theorem 1.Let $\phi(x)$ be the shifted Legendre vector defined in (5), and also suppose $\alpha > 0$ then

$$D^{\alpha}\phi(x);D^{(\alpha)}\phi(x)$$

where $D^{(\alpha)}$ is the $(m+1)(m+1)$ operational matrix of fractional derivative of order α in Caputo sense and is defined as follows:

$$\begin{pmatrix} 0 & 0 & ... & 0 \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ 0 & 0 & ... & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \Theta_{\lceil\alpha\rceil,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \Theta_{\lceil\alpha\rceil,1,k} & ... & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \Theta_{\lceil\alpha\rceil,m,k} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \sum_{k=\lceil\alpha\rceil}^{i} \Theta_{i,0,k} & \sum_{k=\lceil\alpha\rceil}^{i} \Theta_{i,1,k} & ... & \sum_{k=\lceil\alpha\rceil}^{i} \Theta_{i,m,k} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \sum_{k=\lceil\alpha\rceil}^{m} \Theta_{m,0,k} & \sum_{k=\lceil\alpha\rceil}^{m} \Theta_{m,1,k} & ... & \sum_{k=\lceil\alpha\rceil}^{m} \Theta_{m,m,k} \end{pmatrix}$$

Where $\Theta_{i,j,k}$ is given by

$$\Theta_{i,j,k} = 2j+1\sum_{l=0}^{j}\frac{(-1)^{(i+j+k+l)}(i+k)!(l+j)!}{(i-k)!k!\Gamma(k-\alpha+1)(j-l)!(l!)^2(k+l+1-\alpha)}.$$

Proof. The proof is in [14].

Note that in $D^{(\alpha)}$, the first $\lceil\alpha\rceil$ rows, are all zero and if $\alpha = n \in N$, then Theorem 1 gives the same result as(6).

III. SOLUTION OF THE PROBLEM

The factional time derivative telegraph equation of order $\alpha(1<\alpha\le 2)$, is given as

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+\frac{\partial^{\alpha-1}u(x,t)}{\partial t^{\alpha-1}}+u(x,t)=\frac{\partial^{2}u(x,t)}{\partial x^{2}}+f(x,t) \qquad (7)$$

with the initial conditions

$$u(x,0)=g(x),\quad x\in[0,1] \quad and \quad u_t(x,0)=k(x),\quad x\in[0,1] \qquad (8)$$

and Dirichlet boundary condition

$$u(0,t)=h(t),\quad u(1,t)=s(t) \quad 0<t\le 1, \qquad (9)$$

where f,g,k,h and s are known functions, and the function u is unknown. In Eqs.(7-9), the functions f,g,k,h and s generally are not polynomials. We assume that these functions are polynomial or they can be approximated by polynomials to any degree of accuracy. For this purpose, one may use one or two variate Taylor,Chebyshev or Legendre series or other suitable methods. In this article we apply the shifted Legendre method to solve Eqs.(7-9), we approximate $f(x,t),g(x),k(x),h(t)$ and $s(t)$ by (m+1)terms of the shifted Legendre series, thus we get

$$f(x,t);\sum_{i=0}^{m}\sum_{j=0}^{m}f_{i,j}p_i(t)p_j(x)=\phi_m^T(t)F\phi_m(x), \qquad (10)$$

$$g(x);\sum_{j=0}^{m}g_jp_j(x)=G\phi_m(x), \qquad (11)$$

$$k(x);\sum_{j=0}^{m}k_jp_j(x)=K\phi_m(x), \qquad (12)$$

$$h(t);\sum_{i=0}^{m}h_ip_i(t)=\phi_m^T(t)H, \qquad (13)$$

$$s(t);\sum_{i=0}^{m}h_ip_i(t)=\phi_m^T(t)S, \qquad (14)$$

where the shifted Legendre vectors $\phi_m(x)$ and $\phi_m(t)$ are defined similarly to Eq.(5) when $0\le x\le 1$ and $0\le t\le 1$.

$G=[g_0,\dots,g_m],$

$K=[k_0,\dots,k_m],$

$H=[h_0,\dots,h_m]^T,$

$S=[s_0,\dots,s_m]^T,$

$F=[F_0,\dots,Fm], F_i=[f_{0i},\dots,f_{mi}]^T,\quad i=0,1,\dots,m.$

Therefore we consider approximate solution of the form

$$U_m(x,t)=\sum_{i=0}^{m}\sum_{j=0}^{m}u_{i,j}p_i(x)p_j(t)=\phi_m^T(t)U\phi_m(x), \qquad (15)$$

Where

$U=[U_0,\dots,U_m],$

With

$U_i=[u_{0i},\dots,u_{mi}]^T.$

Therefore for finding the numerical approximation solution of $u(x,t)$ we must find the matrix U.The matrix U is an $(m+1)(m+1)$ matrix which contains $(m+1)(m+1)$ unknown coefficients. To find these $(m+1)(m+1)$ unknowns,we should to find $(m+1)(m+1)$ equations. In this section to write equations Eqs.(7-9) in the matrix form. We first consider the first condition of(8)

$$u(x,0)=g(x),\quad x\in[0,1]$$

then we obtain from (11)

$$\phi_m^T(0)U\phi_m(x)=G\phi_m(x),$$

which implies

$$\phi_{n,\tau}^T(0)U=G. \qquad (16)$$

since $\phi_m(x)$ is a basis vector.

Now we consider

$$u_t(x,0)=k(x),\quad x\in[0,1],$$

we have

$$u_t(x,0)=\phi_m(t)^T(0)D^TU\phi_m(x) \qquad (17)$$

Thus from Eqs.(12) and (17) we have

$$K\phi_m(x)=\phi_m^T(0)D^TU\phi_m(x)$$

hence

$$K=\phi_m^T(0)D^TU. \qquad (18)$$

since $\phi_m(x)$ is a basis vector.

Now consider

$$u(0,t)=h(t),\quad 0<t\le 1,$$

from (13) we have

$$\phi_m^T(t)U\phi_m(0)=\phi_m^T(t)H,$$

hence

$$U\phi_m(0)=H. \qquad (19)$$

since $\phi_m^T(t)$ is a basis vector. Then we consider

$$u(1,t)=s(t),\quad 0<t\le 1,$$

from (14) we have

$$\phi_m^T(t)U\phi_m(b)=\phi_m^T(t)S,$$

hence

$$U\phi_m(0)=S. \qquad (20)$$

since $\phi_m^T(t)$ is a basis vector. Finally, we consider Eq.(7), by Eq.(10) and Eq.(15), we obtain

$$\phi_m^T(t)\left(D^T\right)^{\alpha}U\phi_m(x)+\phi_m^T(t)\left(D^T\right)^{\alpha-1}U\phi_m(x)+\phi_m^T(t)U\phi_m(x)$$
$$=\phi_m^T(t)\left(D^T\right)^{2}U\phi_m(x)+\phi_m^T(t)F\phi_m(x)$$

hence the residual $R(x,t)$ for above equation can be written as

$$R(x,t)=\phi_m^T(t)M\phi_m(x),$$

Where

$$M=((D^T)^{\alpha}U+(D^T)^{\alpha-1}U+U-(D^T)^2U-F), \qquad (21)$$

since $\phi_m(t)$ and $\phi_m(x)$ are basic vectors. For find a typical matrix formulation, similar to the typical tau method, we eliminate the two last columns of the matrix E, similarly in our technique we eliminate the two last rows of the matrix E then we generate $(m-1)(m-1)$ linear algebraic equations by using the following algebraic equations:

$$M_{i,j}=0,\quad i=0,\dots,m-1,\ j=0,\dots,m-1. \qquad (22)$$

The number of the unknown coefficients $u_{i,j}$ is equal to $(m+1)(m+1)$. Now we arrange the obtained linear equations to have a system of $(m+1)(m+1)$ equations for the $(m+1)(m+1)$ unknown coefficients. We can find $2m+2$ linear algebraic equations from Eqs.(16) and (18), and $m-1$ linear algebraic equations by choosing $(m-1)$ equations from Eq.(19) and similarly, $(m-1)$ linear algebraic equations by choosing $(m-1)$ equations from Eq. (20) (In this paper we eliminate two first elements in Eqs.(19) and (20), and finally, we generate $(m-1)(m-1)$ linear algebraic equations from Eq.(22). Now $u_m(x,t)$ can be calculated.

IV. NUMERICAL RESULTS

This section is devoted to computational results.We applied the method presented in this paper and solved two examples.

Example1. Consider the following factional time derivative telegraph equation Eq.(7) in the interval $0\le x\le 1$. The initial conditions are given by

$$u(x,0)=0,\quad 0\le x\le 1 \quad and \quad u_t(x,0)=0,\quad 0\le x\le 1 \qquad (23)$$

and subject to the boundary condition

$$u(0,t)=0,\quad u(1,t)=0, \qquad (24)$$

The exact solution,when $\alpha=2$, is

(25)

subject to the boundary condition

$$u(0,t)=t,\quad u(1,t)=1+t, \qquad (26)$$

The exact solution,when $\alpha=2$, is $u(x,t)=(x-x^2)t^2\exp(-t)$ the obtained numerical results by means of the propost method are shown in Table 1 and Fig.1. In Table 1, v compare the exact solution and approximate solution by our method for different values of m. Also values of $u(x)$ for $\alpha=1.7, \alpha=1.8, \alpha=1.9\ and\ \alpha=2$ are shown in Fig.1, we see that as α approaches 2, the numerical solution . to that of integer-order differential equation.

Example 2. Similar to previous examples, consider the fractional telegraph Eq.(7) with $f(x,t)=x^2+t-1$ in the interval $0\le x\le 1$.

$$u(x,0)=x^2, 0\le x\le 1\ and\ u_t(x,0)=1, 0\le x\le 1 \qquad (26)$$

Subject to the boundary condition

$$u(0,t)=t, u(1,t)=1+t \qquad (27)$$

The exact solution, when $\alpha=2$, is

$$u(x,t)=x^2+t \qquad (28)$$

In this example by using just $m=2$, we obtain exact solution such that in [19],we can see best error is 10^{-5}. from Fig. 2, we see that as α approaches 2, the numerical solution converges to that of integer-order differential equation. Also values of $u(x)$ for $\alpha=1.7$, $\alpha=1.8$, $\alpha=1.9$ and $\alpha=2$ are shown in Fig.2.

V. CONCLUSION

The properties of the Legendre polynomials are used to reduce the fractional telegraph equation to the solution of system of nonlinear equations. From the solutions obtained using the suggested method we can conclude that these solutions are in excellent agreement with the already existing ones.

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Table1: Absolute error for α = 2 and different values of m for Example 1.

χ	m = 15	m = 10	m = 8
0	-1.8433910^{-60}	4.685610^{-60}	1.957010^{-60}
0.2	-1.781010^{-24}	-9.046410^{-16}	4.4260010^{-12}
0.4	-1.197810^{-23}	-6.412710^{-16}	-5.084510^{-12}
0.6	-1.197810^{-23}	-6.412710^{-16}	-5.084510^{-12}
0.8	-1.781010^{-24}	-9.046410^{-16}	-4.4260010^{-12}
1	-4.10^{-60}	-5.536510^{-60}	-1.5620510^{-60}

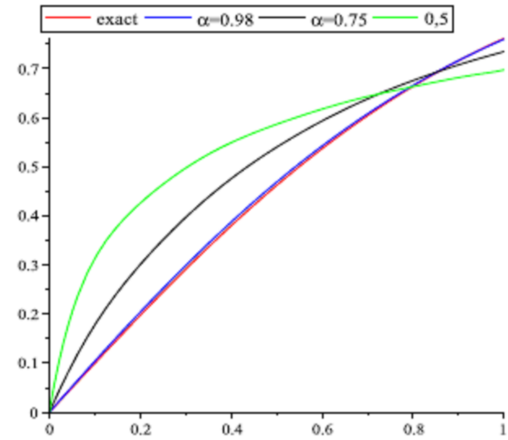

Fig.1. Comparison of $u(x)$ for $m=15$ and with α =1.7,1.8,1.9,2, for Example 1.

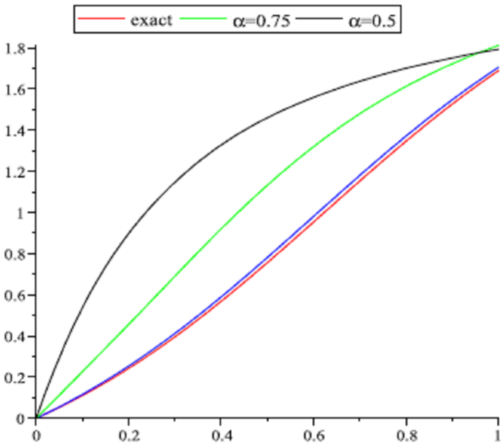

Fig.2. Comparison of $u(x)$ for $m=10$ and with α =1.7, 1.8,1.9,2, for Example 2.